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The family of inverses of operator monotone functions and operator inequalities

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Abstract

Let \mathcal{P}_+ be the set of all non-negative operator monotone functions defined on $[0, \infty)$, and put $\mathcal{P}_+^{-1} = \{h : h^{-1} \in \mathcal{P}_+\}$. Then $\mathcal{P}_+ \cdot \mathcal{P}_+^{-1} \subset \mathcal{P}_+^{-1}$ and $\mathcal{P}_+^{-1} \cdot \mathcal{P}_+^{-1} \subset \mathcal{P}_+^{-1}$. For a function $\tilde{h}(t)$ and a strictly increasing function h we write $\tilde{h} \preceq h$ if $\tilde{h} \circ h^{-1}$ is operator monotone. If $0 \leq \tilde{h} \preceq h$ and $0 \leq \tilde{g} \preceq g$ and if $h \in \mathcal{P}_+^{-1}$ and $g \in \mathcal{P}_+^{-1} \cup \mathcal{P}_+$, then $\tilde{h} \tilde{g} \preceq h g$. We will apply this result to polynomials and operator inequalities. Let $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^m$ be non-increasing sequences, and put $u_+(t) = \prod_{i=1}^n (t - a_i)$ for $t \geq a_1$ and $v_+(t) = \prod_{j=1}^m (t - b_j)$ for $t \geq b_1$. Then $v_+ \preceq u_+$ if $m \leq n$ and $\sum_{i=1}^k b_i \leq \sum_{i=1}^k a_i$ ($1 \leq k \leq m$): in particular, for a sequence $\{p_n\}_{n=0}^\infty$ of orthonormal polynomials, $(p_{n-1})_+ \preceq (p_n)_+$. Suppose $0 < r, p$ and $0 < \alpha \leq \frac{r}{p+r}$. Then $0 \leq A \leq B$ implies $(e^{\frac{rA}{2}} A^s e^{pA} e^{\frac{rA}{2}})^\alpha \leq (e^{\frac{rA}{2}} B^s e^{pB} e^{\frac{rA}{2}})^\alpha$ for $s = 0$ or $1 \leq s \leq 1 + \frac{p}{r}$.

1 Introduction

Let A, B be bounded selfadjoint operators on a Hilbert space. A real-valued (Borel measurable) function $f(t)$ defined on a finite or infinite interval I in \mathbf{R} is called an *operator monotone function* on I and denoted by $f \in P(I)$, provided $A \leq B$ implies $f(A) \leq f(B)$ for every pair A, B whose spectra lie in the interval I . When I is written as $[a, b)$ we simply write $P[a, b)$ instead of $P([a, b))$. It is easy to verify that if a sequence of functions in $P(I)$ converges to f pointwise on I then $f \in P(I)$ and that if $f(t)$ in $P(a, b)$ is continuous

at a from the right then $f(t) \in P[a, b)$. It is well known that t^α ($0 < \alpha \leq 1$), $\log t$ and $\frac{t}{t+\lambda}$ ($\lambda > 0$) are in $P(0, \infty)$. The following Löwner theorem [8] (see also [4, 5]) is essential to the study of this area:

f is operator monotone on an open interval if and only if f has an analytic extension $f(z)$ to the open upper half plane Π_+ so that $f(z)$ maps Π_+ into itself, that is, $f(z)$ is a Pick function.

From this theorem it follows that $f(t)$ in $P(I)$ is constant or strictly increasing and that by Herglotz's theorem $f(t) \in P(0, \infty)$ can be expressed as:

$$f(t) = a + bt + \int_0^\infty \left(-\frac{1}{x+t} + \frac{x}{x^2+1} \right) d\nu(x),$$

where a, b are real constants with $b \geq 0$ and $d\nu$ is a non-negative Borel measure on $[0, \infty)$ satisfying

$$\int_0^\infty \frac{d\nu(x)}{x^2+1} < \infty.$$

Suppose $f \in P[0, \infty)$. Then we can rewrite the above expression as:

$$f(t) = f(0) + bt + \int_0^\infty \left(\frac{1}{x} - \frac{1}{x+t} \right) d\nu(x), \quad (1)$$

For further details on the operator monotone function we refer the reader to Chapter V of [4], [5] and [7].

Thanks to (1), we can see that if $f(t) \geq 0$ is in $P[0, \infty)$, then the holomorphic extension $f(z)$ to Π_+ satisfies $\arg f(z) \leq \arg z$. By making use of this property we can construct operator monotone functions as follows:

if $f(t)$ and $g(t)$ are both positive operator monotone on $(0, \infty)$, then so are $\frac{t}{f(t)}$, $f(t^\alpha)^{1/\alpha}$, $f(t^\alpha)g(t^{1-\alpha})$, $f(t)^\alpha g(t)^{1-\alpha}$ for $0 < \alpha < 1$ and $f(t)g(\frac{t}{f(t)})$; moreover, if f_i is positive operator monotone on $(0, \infty)$ for $1 \leq i \leq n$, then so is $(\prod_{i=1}^n f_i(t))^{1/n}$.

In contrast to this direct way, there is another way to construct an operator monotone function; indeed, it is the way to construct a function whose inverse is operator monotone:

(a) For the function $u(t)$ on $[a_1, \infty)$ defined by

$$u(t) = e^{ct} \prod_{i=1}^k (t - a_i)^{\gamma_i} \quad (c \geq 0, a_1 > a_2 > \cdots > a_k, \quad \gamma_i > 0),$$

if $\gamma_1 \geq 1$, then the inverse function $u^{-1}(s)$ is in $P[0, \infty)$ ([11, 12, 15]):

(b) If $0 \leq f(t)$ is in $P[0, \infty)$, then so is the inverse function of $tf(t)$ ([2]).

Let $\mathcal{P}_+[a, b]$ be the set of all non-negative operator monotone functions defined on $[a, b]$ and $\mathcal{P}_+^{-1}[a, b]$ the set of increasing functions h defined on $[a, b]$ such that the range of h is $[0, \infty)$ and its inverse h^{-1} is operator monotone on $[0, \infty)$. Let g be a non-decreasing function and h a strictly increasing function. Then g is said to be *majorized* by h , in symbols $g \preceq h$, if $g \circ h^{-1}$ is operator monotone on the range of h , where $f \circ g$ means the composite function $f(g)$. In Section 2, we will show that $\mathcal{P}_+[a, b] \cdot \mathcal{P}_+^{-1}[a, b] \subset \mathcal{P}_+^{-1}[a, b]$ and $\mathcal{P}_+^{-1}[a, b] \cdot \mathcal{P}_+^{-1}[a, b] \subset \mathcal{P}_+^{-1}[a, b]$: the first relation includes (a) and (b) given above. Moreover, we will show that if $0 \leq \tilde{h}_i \preceq h_i$ and $0 \leq \tilde{f}_j \preceq f_j$ for $1 \leq i \leq m$, $1 \leq j \leq n$ and if the range of each f_j is $[0, \infty)$, then $\prod_{i=1}^m \tilde{h}_i \prod_{j=1}^n \tilde{f}_j(t) \preceq \prod_{i=1}^m h_i(t) \prod_{j=1}^n f_j(t)$. we will call this the *product theorem*.

Let us define real polynomials $u(t)$ and $v(t)$ by

$$u(t) = \prod_{i=1}^n (t - a_i), \quad v(t) = \prod_{j=1}^m (t - b_j),$$

where $a_i \geq a_{i+1}$ and $b_j \geq b_{j+1}$. The positive and increasing parts of $u(t)$ and $v(t)$ are denoted by $u_+(t)$ and $v_+(t)$ respectively, that is $u_+(t) := u(t)|_{[a_1, \infty)}$ and $v_+(t) := v(t)|_{[b_1, \infty)}$. In Section 3, we will show that $v_+ \preceq u_+$ if $m \leq n$ and $\sum_{i=1}^k b_i \leq \sum_{i=1}^k a_i$ ($k = 1, 2, \dots, m$). In particular, for a sequence $\{p_n\}_{n=0}^\infty$ of orthonormal polynomials with the positive leading coefficients we will show $(p_{n-1})_+ \preceq (p_n)_+$.

Let $h(t) \in \mathcal{P}_+^{-1}[0, \infty)$ and $f_i(t) \in \mathcal{P}_+[0, \infty)$ for $1 \leq i \leq n$, and put $g(t) = \prod_{i=1}^n f_i(t)$. Suppose $0 \leq \tilde{h}(t) \preceq h(t)$. Then, in Section 4, we will show that for the function φ on $[0, \infty)$ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$ for $0 \leq t < \infty$

$$0 \leq A \leq B \implies \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geq \varphi(g(A)^{\frac{1}{2}}h(A)g(A)^{\frac{1}{2}}) = g(A)\tilde{h}(A),$$

which yields a concrete operator inequality that extends the Furuta inequality.

2 $\mathcal{P}_+^{-1}(I)$ and Majorization

Our objectives are non-negative operator monotone functions. A function $f(t)$ that is operator monotone on $-\infty < t < \infty$ is affine, that is, of the form $\alpha t + \beta$. Therefore we confine our attention to operator monotone functions on a subinterval (a, b) of $(-\infty, \infty)$. We consider just the case $-\infty < a$; for, it seems that the case $-\infty < a$ is more useful than the case $b < \infty$. Then every increasing function defined on (a, b) has the right limit at a ; accordingly

we deal with functions defined on $[a, b)$. A “function” means a “continuous function” and “increasing” does “strictly increasing” throughout this paper.

Let us introduce the sets of increasing functions defined on a interval $[a, b)$, although we made a short mention of them in the first section:

Definition 1

$$\mathcal{P}_+[a, b) := \{f | f(t) \geq 0, f(t) \in \mathcal{P}[a, b)\},$$

$$\mathcal{P}_+^{-1}[a, b) := \{h | h(t) \text{ is increasing with the range } [0, \infty), h^{-1} \in \mathcal{P}[0, \infty)\},$$

where h^{-1} stands for the inverse function of h .

It is clear that $\mathcal{P}_+[a, b)$ is a cone and that $(sh^{-1} + (1-s)k^{-1})^{-1} \in \mathcal{P}_+^{-1}[a, b)$ for $0 \leq s \leq 1$ if $h, k \in \mathcal{P}_+^{-1}[a, b)$. We note that $h(t) \in \mathcal{P}_+^{-1}[a, b)$ precisely if $h(t-c) \in \mathcal{P}_+^{-1}[a+c, b+c)$. We give simple examples: $1 \in \mathcal{P}_+[a, b)$ for any interval $[a, b)$, $t^\alpha \in \mathcal{P}_+[0, \infty)$ for $0 < \alpha \leq 1$, $t^\alpha \in \mathcal{P}_+^{-1}[0, \infty)$ for $\alpha \geq 1$ and $(1+t)/(1-t) \in \mathcal{P}_+^{-1}[-1, 1)$, because its inverse function $(t-1)/(t+1)$ belongs to $\mathcal{P}[0, \infty)$.

Definition 2 Let h be non-decreasing function on I and k an increasing function on J . Then we say that h is *majorized* by k , in symbols $h \preceq k$, if $J \subset I$ and $h \circ k^{-1}$ is operator monotone on $k(J)$.

We note that this definition coincides with the following:
for A, B whose spectra lie in J

$$k(A) \leq k(B) \implies h(A) \leq h(B).$$

It is evident that $1 \preceq k$ for every increasing function $k(t)$.

We have the following properties:

- (i) $k^\alpha \preceq k^\beta$ for any increasing function $k(t) \geq 0$ and $0 < \alpha \leq \beta$;
- (ii) $g \preceq h, \quad h \preceq k \implies g \preceq k$;
- (iii) if τ is an increasing function whose range is the domain of k , then
 $h \preceq k \iff h \circ \tau \preceq k \circ \tau$;
- (iv) $k \in \mathcal{P}_+^{-1}[a, b) \iff t \preceq k, \quad k([a, b)) = [0, \infty)$;
- (v) if the range of k is $[0, \infty)$ and $h, k \geq 0$, then
 $h \preceq k \implies h^2 \preceq k^2$;
- (vi) $h \preceq k, \quad k \preceq h \iff h = ck + d$ for real numbers $c > 0, d$.

(i),(ii),(iii) and (iv) are trivial. To see (v) suppose $\phi \in \mathcal{P}_+[0, \infty)$ satisfies $\phi(k(t)) = h(t)$ for $t \in J$ which is the domain of k . Then $\varphi(t) := \phi(\sqrt{t})^2 \in \mathcal{P}_+[0, \infty)$ and $\varphi(k^2) = h^2$; thereby $h^2 \preceq k^2$. To see (vi) assume $h \preceq k$, $k \preceq h$. Then there is ϕ on $k(J)$ such that ϕ and ϕ^{-1} are both operator monotone. Since an operator monotone function is increasing and concave, ϕ must be an increasing linear function. This implies $h(t) = ck(t) + d$ for $c > 0$. The converse is evident.

Lemma 2.1 *If $f(t)$, $g(t)$ and $h(t)$ are all in $\mathcal{P}_+[0, \infty)$, then $h(f(t))g(\frac{t}{f(t)})$ is in $\mathcal{P}_+[0, \infty)$ and so is $f(t)g(\frac{t}{f(t)})$ in particular.*

Proof. It is clear that $h(f(t))g(\frac{t}{f(t)})$ is well-defined on $(0, \infty)$ and has the analytic extension $h(f(z))g(\frac{z}{f(z)})$ to Π_+ . Since $\arg f(z) \leq \arg z$ for $z \in \Pi_+$, we have $\frac{z}{f(z)} \in \Pi_+$ for $z \in \Pi_+$. Hence

$$\arg g\left(\frac{z}{f(z)}\right) \leq \arg \frac{z}{f(z)} < \pi, \quad \arg h(f(z)) \leq \arg f(z) < \pi.$$

Thus $\arg h(f(z))g(\frac{z}{f(z)}) \leq \arg z$, which implies $h(f(z))g(\frac{z}{f(z)})$ is a Pick function. Therefore, $h(f(t))g(\frac{t}{f(t)})$ belongs to $\mathcal{P}_+(0, \infty)$. By considering its continuous extension to $[0, \infty)$ we get $h(f(t))g(\frac{t}{f(t)}) \in \mathcal{P}_+[0, \infty)$; in particular, we get $f(t)g(\frac{t}{f(t)}) \in \mathcal{P}_+[0, \infty)$, although it has been well-known. \square

$\tilde{h} \preceq h$, $\tilde{k} \preceq k$ does not necessarily imply $\tilde{h}\tilde{k} \preceq hk$ even if these functions are all non-negative. For instance, $1 \preceq t$, $t \preceq 1 + t^2$, where all functions are defined on $[0, \infty)$; but $t(1 + t^2) \notin \mathcal{P}_+^{-1}[0, \infty)$ as proved in Example 2.1 of [15], that is to say, $t \preceq t(1 + t^2)$ is false.

The following lemma is beneficial, so we name it.

Lemma 2.2 (Product lemma) *Let $h(t), g(t)$ be non-negative increasing functions defined on $[a, b)$, with $(hg)(a) = 0$, $(hg)(b-0) = \infty$, where $-\infty < a < b \leq \infty$. Then for ψ_1, ψ_2 in $\mathcal{P}_+[0, \infty)$*

$$g \preceq hg \implies h \preceq hg, \quad \psi_1(h)\psi_2(g) \preceq hg.$$

Proof. Define the functions ϕ_i on $[0, \infty)$ by

$$\begin{aligned} \phi_0(h(t)g(t)) &= g(t), & \phi_1(h(t)g(t)) &= h(t), \\ \phi_2(h(t)g(t)) &= \psi_1(h(t))\psi_2(g(t)) \quad (a \leq t < b). \end{aligned}$$

We need to show that if ϕ_0 is operator monotone, then so are both ϕ_1 and ϕ_2 . By putting $s = h(t)g(t)$, ϕ_1 and ϕ_2 can be expressed as

$$\phi_1(s) = \frac{s}{\phi_0(s)}, \quad \phi_2(s) = \psi_1\left(\frac{s}{\phi_0(s)}\right)\psi_2(\phi_0(s)).$$

As we mentioned in the first section, $\phi_1 \in \mathcal{P}_+[0, \infty)$, and $\phi_2 \in \mathcal{P}_+[0, \infty)$ follows from Lemma 2.1. \square

We note that if $\psi_2(t) = t$ in the above lemma, then the last relation implies $\psi_1(h) g \preceq h g$.

Remark 2.1 In the proof of Lemma 2.2, it is crucial that ψ_1 and ψ_2 are in $\mathcal{P}_+[0, \infty)$. For instance, put $h(t) = g(t) = t$ for $0 \leq t < \infty$. Then clearly $g \preceq h g$. Consider the functions $\psi_1(t) = \psi_2(t) = t - 1$ in $\mathcal{P}_+[1, \infty) \cap \mathcal{P}[0, \infty)$. But the function ϕ_2 defined by $\phi_2(t^2) = \psi_1(t)\psi_2(t) = (t - 1)^2$ is not operator monotone even on $[1, \infty)$ (see Section 1 of [11]).

Lemma 2.3 Suppose $a \geq 0$ and $h(t) \in \mathcal{P}_+^{-1}[a, b)$. Then

$$t \cdot h(t) \in \mathcal{P}_+^{-1}[a, b), \quad \text{i.e., } t \preceq th(t).$$

Moreover,

$$h(t) \preceq th(t), \quad t^2 \preceq th(t).$$

The lemma below is a generalization of the statement (b) in Introduction.

Lemma 2.4 Suppose $a \geq 0$. Then

$$\mathcal{P}_+[a, b) \cdot \mathcal{P}_+^{-1}[a, b) \subset \mathcal{P}_+^{-1}[a, b).$$

Moreover, for $f(t) \in \mathcal{P}_+[a, b)$, $h(t) \in \mathcal{P}_+^{-1}[a, b)$ and $\psi(t), \phi(t) \in \mathcal{P}_+[0, \infty)$

$$h \preceq h f, \quad f \preceq h f, \quad \psi(h) \phi(f) \preceq h f.$$

Lemma 2.5 Let $f(t)$ be in $\mathcal{P}_+[0, \infty)$ and $h(t)$ in $\mathcal{P}_+^{-1}[f(0), f(\infty))$. Then

$$t \cdot (h \circ f)(t) \in \mathcal{P}_+^{-1}[0, \infty), \quad t \cdot f \preceq t \cdot (h \circ f).$$

Lemma 2.6 Suppose $a \geq 0$. Then

$$\mathcal{P}_+^{-1}[a, b) \cdot \mathcal{P}_+^{-1}[a, b) \subset \mathcal{P}_+^{-1}[a, b).$$

Moreover, for $h(t), k(t) \in \mathcal{P}_+^{-1}[a, b)$ and for $\psi_1(t), \psi_2(t) \in \mathcal{P}_+[0, \infty)$

$$t \preceq h k, \quad k \preceq h k, \quad \psi_1(h) \psi_2(k) \preceq h k.$$

Lemma 2.7 Suppose $a \geq 0$. Let $h(t) \in \mathcal{P}_+^{-1}[a, b]$ and $f_j(t) \in \mathcal{P}_+[a, b]$ for $1 \leq j \leq n$. Put $g(t) = \prod_{j=1}^n f_j(t)$. Then $h(t)g(t) \in \mathcal{P}_+^{-1}[a, b]$. Moreover, for $\psi_1, \psi_2 \in \mathcal{P}_+[0, \infty)$

$$h \preceq hg, \quad g \preceq hg, \quad \psi_1(h)\psi_2(g) \preceq hg.$$

Remark 2.2 If $\psi_1(t) = h^{-1}(t) - c$ for $c \leq a$, $\psi_1(t) = t$ and $\psi_1(t) = 1$, then $\psi_1(h) = t - c$, $\psi_1(h) = h$ and $\psi_1(h) = 1$, respectively. Similarly, $\psi_2(g)$ may be g or 1 .

So far we have assumed $a \geq 0$ concerning the domain $[a, b]$, but now let us consider the case $a < 0$. For a function $h(t)$ and a real number c , we put $h_c(t) = h(t + c)$. It is evident that $h(t) \in \mathcal{P}_+[a, b]$ if and only if $h_c(t) \in \mathcal{P}_+[a - c, b - c]$, and $h(t) \in \mathcal{P}_+^{-1}[a, b]$ if and only if $h_c(t) \in \mathcal{P}_+^{-1}[a - c, b - c]$, because $h^{-1}(s) = h_c^{-1}(s) + c$ for $0 \leq s < \infty$. By the property (iii), $k(t) \preceq h(t)$ if and only if $k_c(t) \preceq h_c(t)$. Moreover, the function ϕ satisfying $\phi(h(t)) = k(t)$ for $a \leq t < b$ also satisfies $\phi(h_c(t)) = k_c(t)$ for $a - c \leq t < b - c$. Thus we can see Lemma 2.4, Lemma 2.6 and Lemma 2.7 hold even for $a < 0$. We note that $\tilde{h}(t) = t - c$ if $\tilde{h}_c(t) = t$. Thus by Lemma 2.7, we get

Theorem 2.8 (Product theorem) Suppose $-\infty < a < b \leq \infty$. Then

$$\mathcal{P}_+[a, b] \cdot \mathcal{P}_+^{-1}[a, b] \subset \mathcal{P}_+^{-1}[a, b], \quad \mathcal{P}_+^{-1}[a, b] \cdot \mathcal{P}_+^{-1}[a, b] \subset \mathcal{P}_+^{-1}[a, b].$$

Further, let $h_i(t) \in \mathcal{P}_+^{-1}[a, b]$ for $1 \leq i \leq m$, and let $g_j(t)$ be a finite product of functions in $\mathcal{P}_+[a, b]$ for $1 \leq j \leq n$. Then for $\psi_i, \phi_j \in \mathcal{P}_+[0, \infty)$

$$\prod_{i=1}^m h_i(t) \prod_{j=1}^n g_j(t) \in \mathcal{P}_+^{-1}[a, b], \quad \prod_{i=1}^m \psi_i(h_i) \prod_{j=1}^n \phi_j(g_j) \preceq \prod_{i=1}^m h_i \prod_{j=1}^n g_j.$$

Remark 2.3 By Remark 2.2 $\psi_i(h_i)$ may be $t - c_i$ for $c_i \leq a$, h_i or 1 , and $\phi_j(g_j)$ may be g_j or 1 . Hence, for $0 \leq m_1 + m_2 \leq m$, $0 \leq n_1 \leq n$

$$\prod_{i=1}^{m_1} (t - c_i) \prod_{i=1}^{m_2} h_i(t) \prod_{j=1}^{n_1} g_j(t) \preceq \prod_{i=1}^m h_i \prod_{j=1}^n g_j,$$

where we put $\prod_{j=1}^{n_1} g_j(t) = 1$ if $n_1 = 0$.

In the proof of the following theorem, we will use the elementary result that if $h_n(t)$ ($n = 1, 2, \dots$) is continuous and increasing on a finite closed interval I and if $\{h_n\}$ converges pointwise to a continuous function h on I , then $\{h_n\}$ converges uniformly to h on I .

Theorem 2.9 Suppose $h_n(t) \in \mathcal{P}_+^{-1}[a, b)$ ($n = 1, 2, \dots$). If $\{h_n\}$ converges pointwise to a continuous function h on $[a, b)$ and if $h(t) > 0$ for $t > a$ and $h(b-0) = \infty$, then

$$h \in \mathcal{P}_+^{-1}[a, b).$$

Moreover, let \tilde{h}_n ($n = 1, 2, \dots$) be increasing functions on $[a, b)$, and let $\{\tilde{h}_n\}$ converge pointwise to a continuous function \tilde{h} on $[a, b)$. Then

$$\tilde{h}_n \leq h_n \ (n = 1, 2, \dots) \implies \tilde{h} \leq h.$$

In the above theorem, we assumed the continuity of h for the sake of simplicity, though this can be derived by an elementary argument.

Corollary 2.10 Let $h \in \mathcal{P}_+^{-1}[a, b)$, and let g_n , for each n , be a finite product of functions in $\mathcal{P}_+[a, b)$. Suppose $\{g_n\}$ converges pointwise to g on $[a, b)$ such that g is increasing and continuous. Then, for $\psi_1, \psi_2 \in \mathcal{P}_+[0, \infty)$

$$\psi_1(h)\psi_2(g) \leq hg \in \mathcal{P}_+^{-1}[a, b).$$

Proof. By the product theorem $hg_n \in \mathcal{P}_+^{-1}[a, b)$ and $h \leq hg_n$. Thus by Theorem 2.9 $hg \in \mathcal{P}_+^{-1}[a, b)$ and $h \leq hg$. By the product lemma we get $\psi_1(h)\psi_2(g) \leq hg$. \square

3 Polynomials

The aim of this section is to apply the product theorem to real polynomials. (a) in the first section has been shown by analytic extension method in [11]. But its proof was not so easy. To begin with, let us give a simple proof of (a) by using the product theorem:

Another proof of (a). Since $\gamma_1 \geq 1$, $(t - a_1)^{\gamma_1} \in \mathcal{P}_+^{-1}[a_1, \infty)$. For each γ_i ($2 \leq i \leq k$) take a large natural number n to ensure that $n - 1 < \gamma_i \leq n$; then $(t - a_i)^{\gamma_i/n} \in \mathcal{P}_+[a_i, \infty)$. Thus by the product theorem $h(t) := \prod_{i=1}^k (t - a_i)^{\gamma_i} \in \mathcal{P}_+^{-1}[a_1, \infty)$. Besides, $(1 + \frac{c}{n}t) \in \mathcal{P}_+[a_1, \infty)$ for n so that $-\frac{n}{c} \leq a_1$

and $g_n(t) := (1 + \frac{c}{n}t)^n$ converges to e^{ct} for every t , which is increasing and continuous. By Corollary 2.10 $h(t)e^{ct} \in \mathcal{P}_+^{-1}[a_1, \infty)$. \square

The same argument as the above proof leads us to the following:

Proposition 3.1 *Let $h \in \mathcal{P}_+^{-1}[a, b)$ for $-\infty < a < b \leq \infty$. Then for $c \geq 0$ and $\psi_1, \psi_2 \in \mathcal{P}_+[0, \infty)$,*

$$he^{ct} \in \mathcal{P}_+^{-1}[a, b), \quad \psi_1(h)\psi_2(e^{ct}) \preceq he^{ct}.$$

For non-increasing sequences $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^m$, we consider the positive and increasing functions $u(t)$ and $v(t)$ defined by

$$u(t) = \prod_{i=1}^n (t - a_i) \quad (t \geq a_1), \quad v(t) = \prod_{i=1}^m (t - b_i) \quad (t \geq b_1). \quad (2)$$

Lemma 3.2 *Suppose $v \preceq u$ for u and v given in (2). Then $m \leq n$.*

Proof. Since $v \circ u^{-1}(s)$ is concave and non-negative on $0 \leq s < \infty$, $v(u^{-1}(s))/s$ is decreasing. Therefore $v(t)/u(t)$ is decreasing on $a_1 \leq t < \infty$. This implies $m \leq n$. \square

Lemma 3.3 *Let u and v be polynomials defined by (2). Then*

$$m \leq n, \quad b_i \leq a_i \quad (1 \leq i \leq m) \implies v \preceq u.$$

Proof. Consider $t - a_i$ and $t - b_i$ as functions on $[a_1, \infty)$ and $[b_1, \infty)$ respectively. It is evident that $(t - a_1) \in \mathcal{P}_+^{-1}[a_1, \infty)$ and $(t - a_i) \in \mathcal{P}_+[a_1, \infty)$ for every i . Since $\psi(t) := t + (a_i - b_i) \in \mathcal{P}_+[0, \infty)$ and $\psi(t - a_i) = t - b_i$ for $t \geq a_1$, $(t - b_i) \preceq (t - a_i)$ for every i . Hence the product theorem yields $v(t) \preceq u(t)$. \square

The following theorem indicates that the “majorization between sequences” leads us to the “majorization between functions” introduced in the second section.

Theorem 3.4 *Let $u(t)$ and $v(t)$ be polynomials defined by (2). Then*

$$m \leq n, \quad \sum_{i=1}^k b_i \leq \sum_{i=1}^k a_i \quad (1 \leq k \leq m) \implies v \preceq u.$$

We do not know yet if the converse of Theorem 3.4 holds.

Let $\{p_n\}_{n=0}^\infty$ be a sequence of orthonormal polynomials with the positive leading coefficient. It is known that each p_n has n simple zeros $a_1 > a_2 > \dots > a_n$ and there is a zero b_i of p_{n-1} in (a_{i+1}, a_i) . This means $b_i < a_i$ for $i = 1, 2, \dots, n-1$. Thus, by Lemma 3.3 we have

Corollary 3.5 [15] *Let $\{p_n\}_{n=0}^\infty$ be a sequence of orthonormal polynomials with the positive leading coefficient. Denote the restricted part of p_n to $[a, \infty)$ abusively by p_n , where a is the maximum zero of p_n . Then*

$$p_{n-1} \preceq p_n.$$

Now we give a bit of results related to characteristic polynomials of matrices. Let A be a $n \times n$ matrix with singular values $s_1 \geq s_2 \geq \dots \geq s_n$. Then

$$\|A\|_k := \sum_{i=1}^k s_i$$

is called k -norm of A . It is well-known that $\|A\|_k \leq \|B\|_k$ for $k = 1, 2, \dots, n$, if and only if $\|A\| \leq \|B\|$ for every unitarily invariant norm. By using Theorem 3.4 we can easily verify the following:

Corollary 3.6 *Let A, B be $n \times n$ non-negative matrices and p_A, p_B their characteristic polynomials. Then*

$$\|A\|_k \leq \|B\|_k \quad (1 \leq k \leq n) \implies p_A \preceq p_B.$$

We finally deal with a general real polynomial $w(t)$ with imaginary zeros.

Theorem 3.7 *Let $u(t)$ be the polynomial defined in (2) and $w(t)$ the polynomial with imaginary zeros defined by*

$$w(t) = \prod_{j=1}^m (t - \alpha_j) \quad (\Re \alpha_1 \leq t < \infty),$$

where $\Re \alpha_1 \geq \Re \alpha_2 \geq \dots \geq \Re \alpha_m$. Then

$$m \leq n, \quad \sum_{j=1}^k \Re \alpha_j \leq \sum_{j=1}^k a_j \quad (1 \leq k \leq m) \implies w \preceq u.$$

4 Operator Inequalities

In this section we apply the product theorem to operator inequalities. Our interest is inequalities concerning non-negative operators. So we only deal with functions defined on $[0, \infty)$. Let us recall that for $\phi(t) \in \mathcal{P}_+[0, \infty)$

$$\begin{aligned} X^*X \leq 1 &\implies \phi(X^*AX) \geq X^*\phi(A)X \quad (A \geq 0), \\ X^*X \geq 1 &\implies \phi(X^*AX) \leq X^*\phi(A)X \quad (A \geq 0). \end{aligned}$$

The first inequality is called the Hansen-Pedersen inequality [7], from which the second one follows (cf.[13]).

Lemma 4.1 *Let $\phi(t)$ and $f(t)$ be in $\mathcal{P}_+[0, \infty)$. Suppose $h(t)$ and $\tilde{h}(t)$ are non-negative functions on $[0, \infty)$. If $\phi(h(t)f(t)) = \tilde{h}(t)f(t)$, then*

$$0 \leq A \leq B \implies \begin{cases} \phi(f(A)^{\frac{1}{2}}h(B)f(A)^{\frac{1}{2}}) \geq f(A)^{\frac{1}{2}}\tilde{h}(B)f(A)^{\frac{1}{2}}, \\ \phi(f(B)^{\frac{1}{2}}h(A)f(B)^{\frac{1}{2}}) \leq f(B)^{\frac{1}{2}}\tilde{h}(A)f(B)^{\frac{1}{2}}. \end{cases}$$

Proposition 4.2 *Let $h(t) \in \mathcal{P}_+^{-1}[0, \infty)$ and $f_i(t) \in \mathcal{P}_+[0, \infty)$ for $i = 1, 2, \dots$. For each natural number n put $g_n(t) = \prod_{i=1}^n f_i(t)$ and define the function ϕ_n on $[0, \infty)$ by*

$$\phi_n(h(t)g_n(t)) = tg_n(t) \quad (0 \leq t < \infty). \quad (3)$$

Then

$$0 \leq A \leq B \implies \begin{cases} \phi_n(g_n(A)^{\frac{1}{2}}h(B)g_n(A)^{\frac{1}{2}}) \geq Ag_n(A) \\ \phi_n(g_n(B)^{\frac{1}{2}}h(A)g_n(B)^{\frac{1}{2}}) \leq Bg_n(B) \end{cases} \quad (4)$$

Proof. We will show only the first inequality of (4) since the second one can be similarly shown. The product theorem says $\phi_n \in \mathcal{P}_+[0, \infty)$ for every n . By Lemma 4.1

$$\phi_1(g_1(A)^{\frac{1}{2}}h(B)g_1(A)^{\frac{1}{2}}) \geq g_1(A)^{\frac{1}{2}}Bg_1(A)^{\frac{1}{2}} \geq Ag_1(A).$$

Assume (4) holds for n , that is

$$\phi_n(g_n(A)^{\frac{1}{2}}h(B)g_n(A)^{\frac{1}{2}}) \geq Ag_n(A),$$

and denote the larger side (or the smaller side) of this inequality by K (or H). The function ψ_n defined by $\psi_n(tg_n(t)) = f_{n+1}(t)$ is in $\mathcal{P}_+[0, \infty)$, because $f_{n+1}(t) \preceq t \preceq tg_n(t)$. Putting $s = tg_n(t)$, we have

$$\phi_{n+1}(\phi_n^{-1}(s)\psi_n(s)) = s\psi_n(s).$$

Applying Lemma 4.1 to this equality and the inequality $H \leq K$, we get

$$\phi_{n+1}(\psi_n(H)^{\frac{1}{2}} \phi_n^{-1}(K) \psi_n(H)^{\frac{1}{2}}) \geq \psi_n(H)^{\frac{1}{2}} K \psi_n(H)^{\frac{1}{2}} \geq H \psi_n(H).$$

This yields

$$\phi_{n+1}(g_{n+1}(A)^{\frac{1}{2}} h(B) g_{n+1}(A)^{\frac{1}{2}}) \geq A g_n(A) f_{n+1}(A) = A g_{n+1}(A),$$

because $\psi_n(H) = \psi_n(A g_n(A)) = f_{n+1}(A)$ and $\phi_n^{-1}(K) = g_n(A)^{\frac{1}{2}} h(B) g_n(A)^{\frac{1}{2}}$. Thus we have obtained the first required inequality of (4). \square

Remark 4.1 (4) is the generalization of the Furuta inequality [6] (also see [9, 14]):

$$0 \leq A \leq B \implies \begin{cases} (A^{r/2} B^p A^{r/2})^{\frac{1+r}{p+r}} \geq (A^{r/2} A^p A^{r/2})^{\frac{1+r}{p+r}}, \\ (B^{r/2} A^p B^{r/2})^{\frac{1+r}{p+r}} \leq (B^{r/2} B^p B^{r/2})^{\frac{1+r}{p+r}}, \end{cases}$$

where $r > 0$, $p \geq 1$. In fact, let us substitute t^p for $h(t)$ and t^r for $g_n(t)$ in (3), where n is taken as $n-1 < r \leq n$. Since the function ϕ_n satisfies

$$\phi_n(t) = t^{\frac{1+r}{p+r}},$$

(4) deduces the above inequalities.

Proposition 4.3 Let $h(t) \in \mathcal{P}_+^{-1}[0, \infty)$, and let $\tilde{h}(t)$ be a non-negative function on $[0, \infty)$ such that

$$\tilde{h} \preceq h.$$

Let $f_i(t) \in \mathcal{P}_+[0, \infty)$ for $i = 1, 2, \dots$, and put $g_n(t) = \prod_{i=1}^n f_i(t)$. Then for the function φ_n defined by $\varphi_n(h(t)g_n(t)) = \tilde{h}(t)g_n(t)$

$$0 \leq A \leq B \implies \begin{cases} \varphi_n(g_n(A)^{\frac{1}{2}} h(B) g_n(A)^{\frac{1}{2}}) \geq g_n(A)^{\frac{1}{2}} \tilde{h}(B) g_n(A)^{\frac{1}{2}}, \\ \varphi_n(g_n(B)^{\frac{1}{2}} h(A) g_n(B)^{\frac{1}{2}}) \leq g_n(B)^{\frac{1}{2}} \tilde{h}(A) g_n(B)^{\frac{1}{2}}. \end{cases} \quad (5)$$

Theorem 4.4 Let $h(t) \in \mathcal{P}_+^{-1}[0, \infty)$, and let $\tilde{h}(t)$ be a non-negative function on $[0, \infty)$ such that

$$\tilde{h} \preceq h.$$

Let $g_n(t)$ be a finite product of functions in $\mathcal{P}_+[0, \infty)$ for $i = 1, 2, \dots$. Suppose $\{g_n\}$ converges pointwise to g on $[0, \infty)$ such that g is increasing and continuous. Then for the function φ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$

$$0 \leq A \leq B \Rightarrow \begin{cases} \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geq g(A)^{\frac{1}{2}}\tilde{h}(B)g(A)^{\frac{1}{2}}, \\ \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \leq g(B)^{\frac{1}{2}}\tilde{h}(A)g(B)^{\frac{1}{2}}. \end{cases} \quad (6)$$

Furthermore, if $\tilde{h} \in \mathcal{P}_+[0, \infty)$, then

$$0 \leq A \leq B \Rightarrow \begin{cases} \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geq \tilde{h}(A)g(A), \\ \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \leq \tilde{h}(B)g(B). \end{cases} \quad (7)$$

Now we apply Theorem 4.4 to power functions.

Proposition 4.5 Let $h(t) \in \mathcal{P}_+^{-1}[0, \infty)$, and let g be a pointwise limit of $\{g_n\}$, where $g_n(t)$ is a finite product of functions in $\mathcal{P}_+[0, \infty)$. If

$$0 < \alpha < 1, \quad h(t)^\alpha g(t)^{\alpha-1} \preceq h(t),$$

then

$$0 \leq A \leq B \Rightarrow \begin{cases} (g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}})^\alpha \geq g(A)^{\frac{1}{2}}h(B)^\alpha g(B)^{\alpha-1}g(A)^{\frac{1}{2}}, \\ (g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}})^\alpha \leq g(B)^{\frac{1}{2}}h(A)^\alpha g(A)^{\alpha-1}g(B)^{\frac{1}{2}}. \end{cases} \quad (8)$$

Furthermore, if

$$h(t)^\alpha g(t)^{\alpha-1} \in \mathcal{P}_+[0, \infty),$$

then

$$0 \leq A \leq B \Rightarrow \begin{cases} (g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}})^\alpha \geq (h(A)g(A))^\alpha, \\ (g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}})^\alpha \leq (h(B)g(B))^\alpha. \end{cases} \quad (9)$$

Proof. Put $\tilde{h}(t) = h(t)^\alpha g(t)^{\alpha-1}$. Then the assumption means $\tilde{h}(t) \preceq h(t)$. For φ on $[0, \infty)$ defined by

$$\varphi(h(t)g(t)) = \tilde{h}(t)g(t),$$

by Theorem 4.4, we have

$$\varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geq g(A)^{\frac{1}{2}}\tilde{h}(B)g(A)^{\frac{1}{2}}.$$

Since $\varphi(h(t)g(t)) = \tilde{h}(t)g(t) = (h(t)g(t))^\alpha$, or $\varphi(s) = s^\alpha$ for $0 \leq s < \infty$, the above inequality coincides with the first inequality of (8). The rest can be shown in the same fashion. \square

It seems that Proposition 4.5 has numerous applications.

Corollary 4.6 Let a_i, s_i ($i = 1, \dots, n$) and r be real numbers such that $0 = a_0 < a_i$, $0 \leq s_i$ and $0 < r$. Put $s = \sum_{i=0}^n s_i$. If $0 < s \leq 1$, $0 < \alpha \leq 1$, or if

$$1 \leq s_0, r(s - s_0 - 1) \leq s_0, 0 < \alpha \leq \frac{1+r}{s+r},$$

then

$$0 \leq A \leq B \Rightarrow \begin{cases} (A^{\frac{r}{2}} \prod_{i=0}^n (B + a_i)^{s_i} A^{\frac{r}{2}})^\alpha \geq (A^{\frac{r}{2}} \prod_{i=0}^n (A + a_i)^{s_i} A^{\frac{r}{2}})^\alpha, \\ (B^{\frac{r}{2}} \prod_{i=0}^n (B + a_i)^{s_i} B^{\frac{r}{2}})^\alpha \geq (B^{\frac{r}{2}} \prod_{i=0}^n (A + a_i)^{s_i} B^{\frac{r}{2}})^\alpha. \end{cases}$$

We take notice that the Furuta inequality is just the case of $s_0 = p$, $s_i = 0$ for $i \geq 1$ in the above inequalities.

Corollary 4.7 Let g be a pointwise limit of $\{g_n\}$, where $g_n(t)$ is a finite product of functions in $\mathcal{P}_+[0, \infty)$. If $0 < r$, $0 < \alpha \leq \frac{r}{s+r}$, then

$$0 \leq A \leq B \Rightarrow \begin{cases} (g(A)^{\frac{r}{2}} g(B)^s g(A)^{\frac{r}{2}})^\alpha \geq (g(A)^{\frac{r}{2}} g(A)^s g(A)^{\frac{r}{2}})^\alpha, \\ (g(B)^{\frac{r}{2}} g(A)^s g(B)^{\frac{r}{2}})^\alpha \leq (g(B)^{\frac{r}{2}} g(B)^s g(B)^{\frac{r}{2}})^\alpha. \end{cases} \quad (10)$$

The case of $g(t) = e^t$ in (9) has been shown in [3](cf. [10]).

Example 4.1 Let $h(t) = \prod_{i=0}^n (t + a_i)^{s_i}$, where $a_i \geq 0$, $s_i \geq 0$. If $s \leq r$, then

$$0 \leq A \leq B \Rightarrow \begin{cases} |(h(B)e^B)^s (h(A)e^A)^r| \geq (h(A)e^A)^{(s+r)}, \\ |(h(A)e^A)^s (h(B)e^B)^r| \leq (h(B)e^B)^{(s+r)}, \end{cases}$$

where $|X| := (X^*x)^{1/2}$.

Indeed, consider $g(t)$ in the preceding corollary as $\prod_{i=0}^n (t + a_i)^{s_i} e^t$ with $a_i \geq 0$, $s_i \geq 0$, and substitute $2r$ for r and $2s$ for s ; since $1/2 \leq r/(s+r)$ if $s \leq r$, by (10) we get the above inequalities.

Corollary 4.8 Let $h(t) = \prod_{i=0}^n (t + a_i)^{s_i} e^{st}$, where $a_0 = 0$, $a_i > 0$, $s_0 \geq 1$, $s_i \geq 0$, and put $s = \sum_{i=0}^n s_i$. If $0 < r$, $r(s-1) \leq s$, $0 < \alpha \leq \frac{r}{s+r}$, then

$$0 \leq A \leq B \Rightarrow \begin{cases} (e^{\frac{r}{2}A} h(B) e^{\frac{r}{2}A})^\alpha \geq (e^{\frac{r}{2}A} h(A) e^{\frac{r}{2}A})^\alpha, \\ (e^{\frac{r}{2}B} h(B) e^{\frac{r}{2}B})^\alpha \geq (e^{\frac{r}{2}B} h(A) e^{\frac{r}{2}B})^\alpha. \end{cases}$$

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